NPS71-87-001

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Monterey, California

TECHNICAL

NOTES ON SEARCH, DETECTION
AND LOCALIZATION MODELING

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APRIL 1987

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Prepared for:
Naval Postgraduate School
Monterey, CA 93943-5000

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): NPS71-87-001	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION: Naval Postgraduate School; 6b. OFFICE SYMBOL (If applicable): Code 71	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code): Monterey, CA 93943-5000	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING / SPONSORING ORGANIZATION: Naval Postgraduate School; 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code): Monterey, CA 93943-5000	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO / PROJECT NO / TASK NO / WORK UNIT ACCESSION NO

11. TITLE (Include Security Classification)

NOTES ON SEARCH, DETECTION AND LOCALIZATION MODELING

12. PERSONAL AUTHOR(S) Forrest, R. N.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Technical	FROM ______ TO ______	1987 April	73

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	search modeling, detection modeling, localization modeling

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The report is an extended outline of some of the course material used in courses on search, detection and localization modeling.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: R. N. Forrest	22b. TELEPHONE (Include Area Code): (408)646-2653; 22c. OFFICE SYMBOL: Code 55Fo

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

☆ U.S. Government Printing Office: 1986—606-243

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Preface

This report is a collection of material that has been used in courses on search, detection and localization modeling. Its organization follows to some extent material by S. M. Pollock in Selected Methods and Models in Military Operations Research which is listed in the report bibliography. The report is not intended to be a text on these subjects. In particular, in some areas it does not provide the depth of coverage that is found in the book Search and Detection by Alan R. Washburn which is cited in this report as Reference 21.

Table of Contents

I. Detection Models and Signal Detection Theory

Signal detection theory is the basis for analyzing the detection models that are described in this report. In signal detection theory, the decision making portion of a detection system is called the **receiver** and a detection experiment is the observation by a receiver of input data generated during some time interval. The data that is related to a target is called **signal**. The data that is not related to the target is called **noise**. In general, the target data is associated with a localization region that in some cases is called a **resolution cell**. When a detection experiment is performed, either the event H_0 = **{the receiver input is noise}** or its complement H_1 = **{the receiver input is signal and noise}** will occur. In the first detection models that are described here, after analysis of the input data by a receiver, either the event D_0 =**{the receiver decides the input is noise}** or its complement D_1 = **{the receiver decides the input is signal and noise}** will occur. Detection models for which D_1 is the complement of D_0 are called binary detection models or forced choice detection models. Four events which are important in binary detection model are indicated in the Venn diagram of Figure 1.

The Venn diagram emphasizes a decision problem that is associated with a receiver that can be modeled using a binary detection model. The problem is this: Under what conditions should the event D_1 occur? That is, under what conditions should a receiver decide that the input data indicates a target was present in an observed localization region either at the time or prior to the time of the observation?

	H_0	H_1
D_0	$D_0 \cap H_0$	$D_0 \cap H_1$
D_1	$D_1 \cap H_0$	$D_1 \cap H_1$

Figure 1. Four events of importance in binary detection models.

In the detection model descriptions that follow, the following notation and terminology are used: $p_f = P(D_1|H_0)$, the probability of D_1 given H_0, is called the **false alarm probability**; $p_d = P(D_1|H_1)$, the probability of D_1 given H_1, is called the **detection probability** and $P = P(H_1)$, the probability of H_1, is called the **prior probability**. It is the probability that a target will be in the localization region when the input data is generated.

In the detection models, the input to a receiver consists of a sequence $y_1,...y_m$ that is a realization of a discrete parameter stochastic process whose random variables are elements of a continuous parameter stochastic process. Although the receiver input in some cases can be considered to be described by a continuous stochastic process, because of the finite amount of information (unique data) contained in a bounded sequence of a continuous stochastic process, because of the finite amount of information (unique data) contained in a bounded sequence of finite length, a discrete stochastic process is sufficient to represent the input in these cases. This is established formally by the stochastic sampling theorem. In the models, for an observation, the input stochastic process has the following characteristics: It is a noise process when there is no target data and it is a noise process plus a signal process when there is target data. In the first detection model that is described in Section III, the signal process is a deterministic process. In the second and third detection models that are described there, the signal process is a random process. To specify the noise process or a random signal process, one needs only to specify the joint distribution of the finite sequence of random variables that determine the process. If the signal process is a deterministic process, the signal values are a sequence of values that are assumed to be known before an observation is performed. Consequently, in this case, to specify the process, one needs only to specify these values.

II. Decision Criteria

To simplify the discussion of decision criteria and decision rules, a receiver's input will be assumed to be determined by a single decision random variable Y. In this case, the input process in determined by the conditional distribution function $F_Y(y|H_0)$ when the input is noise alone and by the conditional distribution function $F_Y(y|H_1)$ when the input is signal plus noise.

The condition that a receiver's input is required to satisfy in order that the event D_1 will occur can be specified in terms of a decision rule. For the assumed case, a decision rule is a rule which determines for every observable value of Y the decision that the receiver is to make. The decision rule can be considered to be a function $\phi(y)$ which relates each observable value y to one or the other of the following two commands: d_0 = "Decide that the receiver input was noise." and d_1 = "Decide the input was signal and noise.". Choosing a decision rule $\phi(y)$ defines a set Ω of observable values of Y such that the event $D_1 = \{ Y \epsilon \Omega \}$.

The problem which was considered in Section I can now be restated in the following way: What criterion should be adopted in order to determine a decision rule or, equivalently, its corresponding set Ω ? A desirable characteristic for a criterion is suggested by the following argument: Consider the odds in favor of H_1 given y is observed. That is, consider

$P(H_1|Y = y)/P(H_0|Y = y)$. One might expect that y would be a member of the set Ω if and only if y made this ratio equal to or greater than some value k. But this is equivalent to defining Ω as follows: $\Omega = \{ y : L(y) \geq K \}$ where $L(y)$ is the likelihood ratio associated with an observed value y and K is a constant related to the constant k. This suggests that choosing an optimum criterion is equivalent to choosing an optimum value for K. Four specific decision criteria are defined next in terms of K. For each criterion, Ω has the above form. But for each criterion the choice of K is different. The decision criteria are:

1. **The Neyman-Pearson Criterion:** Choose Ω so that p_d is a maximum subject to the constraint that $p_f \leq \alpha$ where α is a specified value. For a continuous decision random variable, the constant K is chosen so that $p_f = \alpha$.

2. **The Bayes Criterion:** Choose Ω so that the expected cost of a receiver's decision is a minimum. For a continuous decision random variable, $K = [(c_{10}-c_{00})/(c_{01}-c_{11})](1-P)/P$ if $c_{10} > c_{00}$ and $c_{01} > c_{11}$ where c_{ij} is the cost of $D_i \cap H_j$.

3. **The Ideal Observer Criterion:** Choose Ω so that the probability that the receiver makes an incorrect decision is a minimum. $K = (1-P)/P$ for a continuous decision random variable.

4. **The Minimax Criterion:** Choose Ω when P is unknown so that the maximum expected cost of a receiver's decision is a minimum. If $c_{10} > c_{00}$ and $c_{01} > c_{11}$, then

$K = [(c_{10}-c_{00})/(c_{01}-c_{11})](1-P^*)/P^*$ for a continuous decision random variable. Here, P^* is the value of the prior probability P which would make the expected cost of a receiver's decision a minimum for this choice of K.

If a model which specifies the conditional distributions $F_Y(y|H_0)$ and $F_Y(y|H_1)$ and a decision rule are adopted, then the value of p_f and the value of p_d are determined. This pair of values (p_f,p_d) is called a receiver operating point. If the decision rule results from using a likelihood ratio criterion such as one of the four listed above, then it will involve the parameter K through the relation $\Omega = \{y: L(y) \geq K\}$. And, for a given value of K, since Ω uniquely determines the pair (p_f,p_d), a single operating point results. By varying K, a set of operating points can be generated which determines a **receiver operating characteristic curve** or **ROC curve**. Different ROC curves can be produced by changing either one or both of the conditional distributions which implies changing either the signal process or the noise process.

A decision rule which results from using a likelihood ratio criterion in a model in which the input process is determined by a set of m random variables can be expressed in terms of a set Ω as follows: $\Omega = \{ (y_1,...,y_m) : L(y_1,...,y_m) \geq K \}$ where K is specified in the same way as in the case in which $m = 1$.

III. Three Binary Detection Models

Three detection models are examined in this section. For the first two detection models, the input stochastic process for an observation is defined by a time sequence of continuous random variables. The random variables represent a sample from a continuous parameter stochastic process which is sampled at times such that the random variables are independent. For the third detection model, the input stochastic process is a counting process and it is defined by a single discrete random variable that is equal to the number of events that are counted during the observation.

Model I: In the first detection model, a sampled noise value is a value of a normally distributed random variable with mean zero and with known variance σ^2 and a sampled signal value is a known value of a deterministic variable. Thus, the input process corresponding to an observation consists of some number m of independent normal random variables $Y_1,...,Y_m$ each with variance σ^2. And, for $i = 1,2,3, \cdots, m$, when a signal is not present the mean of Y_i is zero and when a signal is present the mean is s_i. The result of using a likelihood ratio decision rule in the model can be expressed in terms of a random variable Z. This random variable is called a crosscorrelation statistic and it is defined by $Z = \Sigma\, s_i \cdot Y_i$ where the sum index $1 = 1,2, \cdots ,m$ here and in the remainder of this section. However, it is more convenient to express the result in terms of a statistic V which is defined by $V = Z/\sigma_Z$. In terms of this random variable, the conditional probabilities p_f and p_d are given by: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$ where Φ symbolizes the standard normal cumulative distribution **function**, the threshold $v^* = (1/\sigma_Z)(\sigma^2 \ln K + (1/2)\ \Sigma\ s_i^2)$ and $d = \Sigma\ s_i^2/\sigma^2$. The parameter d is called the **detection index.**

Often, the input stochastic process represents a quantity whose square is proportional to power. In such a case, the average receiver input power is the random variable $\Sigma\ Y_i^2/m$. If a signal is not present, the expected average receiver input power is $N = \Sigma\ \sigma^2/m = \sigma^2$ where N is called the **noise**. The average receiver input signal power is $S = \Sigma\ s_i^2/m$ where S is called the **signal**. In these terms, $d = m\cdot(S/N)$ where S/N is called the **signal-to-noise ratio**.

If a receiver's input can be considered to be a time sequence of continuous voltage values such as in the case of a sonar receiver, in some cases a frequency representation can be used that involves the concept of receiver bandwidth. In these cases, the noise process is assumed to be such that $m = t/\delta t$ where t is the **integration time** (the duration of an observation) and δt is the **time between samples** with $\delta t = 1/[2(BW)]$ where BW is the **bandwidth** and δt is determined by the sampling theorem. This implies that the detection index can be written as $d = 2t\cdot(BW)(S/N)$. By defining N_0 as the **power spectral density** where $N_0 = N/BW$, the detection index can also be written as $d = 2t\cdot(S/N_0)$. In Reference 2, the conditions required by the first model are called Case I and in the following sections the first model is called the Case I model. A receiver that processes data such that it would implement a likelihood ratio decision rule under the conditions of the first model is called a crosscorrelation detector. If the description of the input noise is adequate, a Case I model can be used to obtain an estimate of an upper bound on a detection system's performance, since all the information necessary to define the signal is assumed to be known.

Model II: In the second detection model, a sampled noise value is a value of an independent normal random variable with mean zero and with variance σ^2. And a sampled signal value is an independent random variable with mean zero but with variance σ_S^2. Thus, the input process corresponding to an observation consists of some number m of independent normal random variables $Y_1,\ldots,Y_m$ each with mean zero and each with variance σ^2 when a signal is not present and with variance $\sigma^2 + \sigma_S^2$ when a signal is present. The result of applying a likelihood ratio decision rule in this model can be expressed in terms of a statistic X which is defined by $X = \Sigma\, Y_i^2$. When a signal is not present, the statistic X/N has a chi-square distribution with m degrees of freedom. When a signal is present, the statistic $X/(N+S)$ has a chi-square distribution with m degrees of freedom. So, in terms of these statistics, the conditional probabilities p_f and p_d are defined as follows:

$p_f = P(X_m^2 \geq x^*/N)$ and $p_d = P\{X_m^2 \geq (x^*/N)[1/(1+S/N)]\}$

where X_m^2 is a chi-square random variable with m degrees of freedom, x^* is a number which is determined by the decision rule and S/N is the signal-to-noise ratio. A receiver that would implement a likelihood ratio decision rule under the conditions of the second model is called an energy detector (or square law detector).

The mean of a chi-square random variable with m degrees of freedom is m and the variance is $2m$. By the central limit theorem, as the number of degrees of freedom of a chi-square random variable becomes large it can be approximated by a normal random variable with the same mean and variance. Hence, for a sufficiently large sample size m or equivalently, for a sufficiently large time bandwidth product $t\cdot(BW)$, the conditional probabilities p_f and p_d can be approximated by $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi\{[1/(1+S/N)](v^* - d^{\frac{1}{2}})\}$ where the threshold value $v^* = (x^* - m\sigma^2)/[(2m)^{\frac{1}{2}}\sigma^2]$ and $d = t\cdot(BW)(S/N)^2$ if the concept of bandwidth is applicable. If, in addition, the noise N is significantly larger than the signal S, that is, if $t\cdot(BW) >> 1$ and $S/N << 1$, then p_f and p_d can be approximated by: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$. In Reference 2, the conditions required for this approximation are called Case II and in the following sections the limiting form of the second model is called the Case II model. In Reference 2, an example is given that uses the Case II model to estimate the performance of a passive sonar system.

Model III: In the third detection model, a sampled noise value and a sampled signal value are values of independent random variables that are determined by independent Poisson processes observed for a time interval t. The noise process is characterized by a counting rate α, the signal process is characterized by a counting rate α_s and the noise and signal processes are additive. Consequently, when the input is noise alone, the input is the value of a random variable Y that has a Poisson distribution with parameter αt. And, when a target is not present, the input is a value of a random variable that has a Poisson distribution with parameter $(\alpha + \alpha_s)\cdot t$. For a likelihood ratio decision rule, $p_f = 1 - P(y^*;\alpha t)$ and $p_d = 1 - P[y^*;(\alpha + \alpha_s)\cdot t]$ where $P(y;\theta)$ symbolizes the cumulative Poisson distribution function with parameter θ and y^* is a threshold value of y determined by the decision rule. In terms of the above notation, both the mean and variance of a Poisson random variable are equal to θ. When θ is large, the cumulative Poisson distribution function can be approximated by the cumulative distribution function of a normal random variable having the same mean and variance. Using this approximation for cases where αt is large gives $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi\{[1/(1 + \alpha_s/\alpha)^{\frac{1}{2}}](v^* - d^{\frac{1}{2}})\}$ where $v^* = (y^* - \alpha t)/(\alpha t)^{\frac{1}{2}}$ and $d = \alpha\cdot t\cdot(\alpha_s/\alpha)^2$. In addition, if the signal counting rate is significantly smaller than the noise counting rate, that is, if $\boldsymbol{\alpha\cdot t >> 1}$ and $\boldsymbol{\alpha_s/\alpha} << 1$, then p_f and p_d can be approximated by: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$.

This detection model might be used to describe a receiver whose input for an observation is the number of photons counted by a radiation detector in situations where αt, the expected number of counts when no signal is present, is of the order of thirty or more. In this case, the target might be a wake segment for example.

When a likelihood ratio decision rule is used in the three models discussed above, for the first model and under limiting conditions for the second and third models, the following result is obtained: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$ where the definition of v^* depends on the noise power N for the first and second models. For a sonar receiver described by the first model, that is, by the Case I model: $d = 2t \cdot (BW)(S/N)$. For a sonar receiver described under the limiting conditions for the second model, that is, by the Case II model, $d = t \cdot (BW)(S/N)^2$. So, in either a Case I model or a Case II model of a sonar receiver, the detection index d is a function of the time bandwidth product $t \cdot (BW)$ and signal-to-noise ratio S/N. Since sonar equations relate S/N to system, target and environmental parameters, a sonar equation can be used to relate S/N to these parameters in a model of a sonar receiver.

IV. General Detection Models

The detection models that have been considered to this point are based on binary detection theory. After each observation, a receiver decides either that the input corresponding to the observation was noise or else it decides it was signal plus noise. However, in some detection systems this decision is delayed. In a computational sense, a model of such a detection system is generally complex relative to a binary detection model. To illustrate this, consider an active sonar system whose receiver includes an operator. Suppose the probability that the operator will detect a target echo has been determined in a laboratory experiment in which the operator was required to decide after each input corresponding to a resolution cell that either the input was a target echo (signal) and noise or the input was noise alone. In addition, suppose that under operational conditions the operator normally delays this decision. Then, in general, the probability that the operator will decide that the input corresponding to a resolution cell that contains a target is a target echo and noise will not be equal to the probability of the event in the forced choice experiment. And, in addition, the probability that the operator will decide the input corresponding to a resolution cell that does not contain a target is a target echo and noise will not be equal to the probability of this event in the forced choice experiment. Consequently, in general, the value of both p_d and

p_f for an operational environment will be different than that for the laboratory environment.

One model that has been proposed to deal with this kind of situation defines the event that a receiver decides that the input corresponding to a resolution cell is signal and noise to be equivalent to the event that out of n consecutive observations at least k of them would result in the decision that the input was signal and noise in a forced choice experiment. The model is said to be based on an k-out-of-n detection criterion. With this criterion, the probability that a target will be first detected on the j^{th} observation can be found as follows: Determine the 2^j sequences of forced choice responses that could result for a sequence of j consecutive observations. Next, determine the probability of occurrence for each sequence that first satisfies the k-out-of-n detection criterion on the j^{th} observation. The probability of first detection on the j^{th} observation is equal to the sum of these probabilities. The cumulative probability of detection at the j^{th} observation is the sum of the probabilities of first detection on the i^{th} observation for $i = 1,2, \cdots ,j$.

V. Signal-to-Noise Ratio Detection Models

In some radar and sonar detection models, for a specified value of p_f, a minimum acceptable value of p_d is defined. This minimum acceptable value of p_d and the specified value of p_f define what can be called a minimum acceptable signal-to-noise ratio $(S/N)_m$ if p_d is a nondecreasing function of S/N. In some sonar detection models, this signal-to-noise ratio in decibels is called the **detection threshold** DT. In symbols, $DT = 10 \log(S/N)_m$. If the minimum acceptable value of p_d is .5, then DT is usually called the **recognition differential** RD. The difference between the signal-to-noise ratio in decibels and RD (or DT) is called the **signal excess** SE. In symbols, $SE = 10 \log(S/N) - RD$.

One interpretation of signal excess is that for a localization region containing a target detection occurs with probability one if $SE \geq 0$ and with probability zero if $SE < 0$. This interpretation provides the basis for defining detection in the three encounter detection models that are discussed in the Section VII. A more consistent interpretation of signal excess is: If $SE \geq 0$, then the probability of detection p_d is greater than or equal to the minimum acceptable value (.5 if recognition differential RD is used to define signal excess). For cases where p_d increases rapidly with signal excess in the neighborhood of zero signal excess, the two interpretations are essentially equivalent. For a discussion of this point as well as a discussion of an operational case in which receiver decisions are delayed, see Reference 3.

Signal excess (signal-to-noise ratio) models provide a basis for general detection models, in particular, models that describe nonstationary noise and signal processes and randomly changing decision rules. This is illustrated by the models described in Section VII. In addition, signal excess models provide a basis for delayed receiver decision models. This is illustrated by the active sonar detection models in both Reference 4 and Reference 5 that are based on a k-out-of-n detection criterion. In all of these models, the signal-to-noise ratio and the recognition differential are random variables.

Using $X(t)$ to represent a random variable corresponding to an index time t and a subscript to identify the random variable, for a passive sonar receiver, the signal-to-noise ratio in decibels associated with a decision at the index time can be expressed as follows: $X_{SL}(t) - X_{TL}(t) - [X_{NL}(t) - X_{DI}(t)]$. In this expression, SL represents source level, TL represents transmission loss, NL represents noise level and DI represents directivity index. Since signal excess SE is defined to be the difference in decibels between the signal-to-noise ratio and the recognition differential (or detection threshold), it too is a random variable and, for any decision time t, one can write:

(1) $X_{SE}(t) = X_{SL}(t) - X_{TL}(t) - [X_{NL}(t) - X_{DI}(t)] - X_{RD}(t)$.

The distributions of the random variables on the right side of Equation 1 determine the distribution of the signal excess. In the passive sonar detection model described in Reference 6, $X_{SL}(t)$, $X_{RD}(t)$ and, in effect, $X_{NL}(t)$ are normally distributed random variables while $X_{TL}(t)$ is a uniformly distributed random variable. In the three signal excess models that are described in Section VII, all of the random variables in Equation 1 are normally distributed.

It is sometimes convenient to write Equation 1 as follows:

(2) $X_{SE}(t) = SE(t) + X(t)$

In Equation 2, $SE(t)$ is the expected value of the signal excess determined by the following expected value equation:

(3) $SE(t) = SL(t) - TL(t) - [NL(t) - DI(t)] - RD(t)$

where each term on the right represents the expected value of the indicated random variable and $X(t)$ is a random variable that determines the stochastic character of the signal excess. Since $SE(t)$ is the mean of $X_{SE}(t)$; by Equation 2, the mean of $X(t)$ is equal to zero and the standard deviation of $X(t)$ is equal to the standard deviation of $X_{SE}(t)$. If σ represents the standard deviation of $X_{SE}(t)$ and the random variables on the right side of Equation 1 are statistically independent, then $\sigma^2 = \sigma_{SL}^2 + \sigma_{TL}^2 + \sigma_{NL}^2 + \sigma_{DI}^2 + \sigma_{RD}^2$. This relation has been used to determine a standard deviation for the signal excess in operational models.

VI. General Encounter Models

A basic problem associated with search modeling is that of determining the probability that a target will be detected by at least one detection system in an encounter with one or more detection systems on or before the n^{th} decision in the encounter has occurred regarding the presence or absence of a target in a localization region. In the encounter models that are considered in this report, during a search, observations are made of a series of localization regions. For localization regions that do not contain a target, the signal observed during the observation of the region is zero. In addition, in these models, the time to resolve a false alarm is zero. However, p_d and p_f are considered to be determined by some criterion such that p_f is less than one. Consequently, although the time to resolve a false alarm is zero, the cost associated with a false alarm is not zero.

Using an order number rather than a time to index a decision and a random variable N to represent the decision order number at which detection first occurs, the cumulative probability of detection for an encounter is given by: $P(N \le n) = \Sigma\, P(N = i)$ where the sum index $i = 1,2, \cdots ,n$. It also given by: $P(N \le n) = 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_n)$ where $g_i = P(N = i | \overline{N \le i-1})$ is the probability of the event detection on the i^{th} decision conditioned on the event no detection on an earlier observation. The second expression for $P(N \le n)$ is generally of greater interest than the first expression, since

g_i can usually be more directly related to operational parameters such as target range and target environment that determine a targets detectability than can $P(N = i)$. If a target is not present in the localization region that corresponds to the i^{th} decision, then g_i will be zero. The second expression above for $P(N \le n)$ can also be written in the following form: $P(N \le n) = 1 - \exp[\Sigma \ln(1 - g_i)]$ where the sum index $i = 1,2, \cdots ,m$. In this expression, $\ln(1 - g_i)$ can be approximated by $-g_i$ for $g_i << 1$. With this approximation, $P(N \le n) = 1 - \exp(-\Sigma g_i)$. This expression provides a basis for developing a continuous analog for $P(N \le n)$ that has been used to describe detection systems for which decisions can be considered to occur continuously. That is, for cases where the time of the observations corresponding to a decision and the time between decisions are both negligible relative to the average time of an encounter. The analog can be developed as follows: First, replace the decision index by a time index. Next, let δt be the time between decisions. Then, the cumulative probability of detection after the n^{th} decision can be expressed by: $P(N \le n) = P(T \le t)$ where $T = N \cdot \delta t$ and $t = n \cdot \delta t$. Next, define a detection rate function (a probability of detection per unit time) by the expression: $(1/\delta t)P[T = t|\overline{T \le (n-1) \cdot \delta t}]$. If T is considered to be a continuous random variable, then this expression can be used as a guide for defining a continuous detection rate $\tau(t)$. The definition is:

$$(4) \qquad \tau(t) = \lim\{(1/\delta t)P(t < T \le t+\delta t \mid \overline{T \le t})\}$$

where the limit is for δt approaching zero and $P(T \le t)$ is such that $P(T \le t) = P(N \le n)$ when t corresponds to n for all values of n that are involved and $P(T \le t)$ is a nondecreasing function of t. Equation 4 implies the following differential equation: $dp(t)/dt = [1 - p(t)]\tau(t)$ where $p(t) = P(T \le t)$. The solution to this differential equation is:

$$(5) \qquad P(T \le t1) = 1 - P(T \le t0) \exp[-\int_{t0}^{t1} \tau(t)dt]$$

where t is the time index for a decision during an encounter, t0 and t1 are times of the encounter and $t1 > t0$. A $\tau(t)$ that is based on a visual detection model is described in Reference 7. If the detection capability of a detection system is assumed to depend on a target's position relative to the detection system during an encounter but not to depend on the clock time, then the time index of a decision can be a relative index that determines a target position that is associated with a decision rather than the clock time associated with the decision.

The above results apply to the case of an encounter between a target and a detection system consisting of more than one sensor. However, if the sensors are not located on a single platform or at a single point, it is generally more convenient to describe encounters of this kind in terms of the encounters between the target and the individual sensors. In either case, if the event that a sensor detects a target during an encounter is not independent of the this event for other sensors, in order to describe this in an encounter model, the correlation between this sensor and the other sensors must be specified. This has been done in some models as follows: First determine the probability of detection during the encounter for each sensor involved. Then determine the probability that at least one of the sensors will detect the target during the encounter for the case where the random factors that determine detection are independent and again for the case where they are completely dependent. Let P_i be the probability that the i^{th} sensor detects the target during the encounter and let the probability that at least one sensor detects the target be P_I in the first case and P_D in the second case. For the first case, $P_I = 1 - (1 - P_1)(1 - P_2) \cdots (1 - P_n)$ where n is the number of sensors involved. And, for the second case, $P_D = P_m$ where $P_m \geq P_i$ for $i = 1,2, \cdots ,n$. Next, combine the two probabilities as follows: $P = \alpha \cdot P_D + (1 - \alpha) \cdot P_I$. A way to determine a value for α is described in Reference 8.

VII. Three Signal Excess Encounter Models

In the three models described in this section, detection is defined in terms of signal excess as it is in Section V. Each of the models determines a cumulative probability of detection for a target in an encounter with a passive sonar system. An observation in the models is indexed by time and the index can usually be considered to be the time at the end of the observation. During an encounter, observations are made of a series of localization regions. For localization regions that do not contain a target, the signal observed during the observation of the region is zero. The time to resolve a false alarm is zero. However, false alarms are not ignored in that the value of RD (or DT) is determined by some specified false alarm probability. Consequently, although the time to resolve a false alarm is zero, the cost associated with a false alarm is not zero. In determining signal excess in the models, it is convenient to use Equation 2. For each decision in an encounter, there is a random variable $X(t)$ defined by Equation 2 that determines the random character of the signal excess. For a sequence of decisions, the set of these random variables ordered by their time index constitutes a stochastic process. The joint distributions of these random variables determines the nature of this process. In the three encounter detection models described in this section, the process is called a lambda sigma jump process. The jumps in the series occur at times determined by a Poisson process with a mean rate lambda. This implies that the time between jumps is a random variable with an exponential distribution and that the expected times between jumps τ is equal to the reciprocal of lambda. The time series that are generated by lambda sigma jump processes are represented by Figure 2 below.

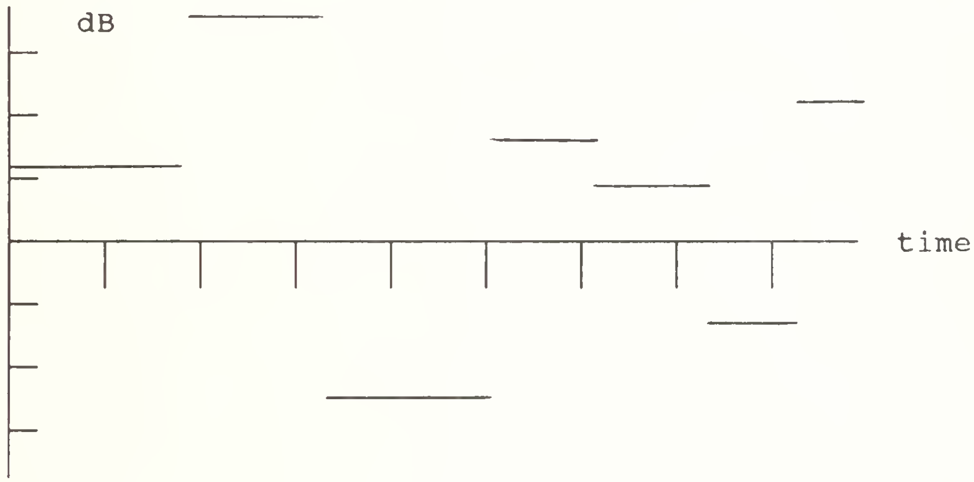

Figure 2. A time series representing a realization of a lambda sigma jump process. On the plot, σ in dB equals one unit on the vertical axis and τ equals one time unit on the horizontal axis.

From Figure 2, note that the observed values of neighboring random variables are equal unless a jump has occurred between them. When a jump occurs, the first random variable after the jump is normally distributed with mean zero and variance σ^2 and it is independent of all the random variables before the jump. Conditioned on a jump pattern, this random variable and all the random variables between it and the next jump are dependent and the correlation coefficient between any pair is one. That is, if the value of the signal excess is known at some time, then all of the values between the last jump before that time and the first jump after that time are also known. However, since the jumps occur randomly, knowing the value of the signal excess with certainty at some time does not determine the values of the signal excess with certainty at neighboring times. In the unconditioned case, the correlation coefficient between the random variables $X(t)$ and $X(t+\tau)$ is equal to $1/e$. For this reason, τ is referred to as a relaxation time. If only the transmission loss is considered to be determined by a stochastic process, then only the transmission loss will be a random quantity and τ is the expected time between jumps in the random component of the transmission loss. It appears that the use of the lambda sigma jump process is based more on past practice than on experimental justification. In this regard, see Reference 9. By referring to Equation 1, it can be seen that the lambda sigma jump stochastic process is equal to the sum of the stochastic processes that determine the random variables on the right side of this equation. But, the nature of the processes should depend on the encounter that is being modeled. This suggests that an adequate description of an encounter using a lambda sigma jump process might not be possible in all cases.

In the three encounter models described below, detection is defined in terms of signal excess as described in Section IV and decisions are indexed by a time that can usually be considered to be the time of the decision. During an encounter, observations are made of a series of localization regions. For localization regions that do not contain a target, the signal observed during an observation of the region is zero. In the models, the time to resolve a false alarm is zero. However, false alarms are not ignored in that the value of RD (or DT) is determined by some specified false alarm probability. Consequently, although the time to resolve a false alarm is zero, the cost associated with a false alarm is not zero.

The First Passive Sonar Encounter Detection Model: This model describes an encounter as a series of decisions. In the model, the integration time that determines the recognition differential is equal to the duration of the observations corresponding to a decision. For an encounter in which $SE(t)$ is unimodal and in which the time of the single maximum is prior to the end of the encounter, it is shown in Reference 10 that the probability that detection will occur during the encounter is given by the following equation:

(6) $p = 1 - [(1 - p_c)/(1 - \beta \cdot p_c)](1 - \beta \cdot p_1) \cdots (1 - \beta \cdot p_m)$

where $\beta = 1 - \exp(-\delta t/\tau)$ and $p_i = \Phi[SE(t_i)/\sigma]$ for $1 = 1,2,3, \cdots ,m$. Here, δt indicates the duration of an observation and Φ indicates the standard normal cumulative distribution function as before. The integer c is the index of the decision time t_c for which $SE(t_c)$ is greater than or equal to $SE(t_i)$ for any time t_i and $t_c \leq t_m$.

As τ approaches zero, β approaches one and Equation 6 approaches this form:

(7) $p = 1 - (1 - p_i) \cdots (1 - p_m)$.

In this limit, the signal excess random variables are all independent. Note that Equation 7 applies without the condition that SE(t) be unimodal.

As τ approaches infinity, β approaches zero and Equation 6 approaches this form:

(8) $p = p_c$.

In this limit, the correlation coefficient between any pair of signal excess random variables is equal to one. Note that Equation 8 applies without the condition that SE(t) be unimodal. Equation 8 is said to define a complete dependence encounter model.

The Second Passive Sonar Encounter Detection Model: This model is in a sense a third limiting form of the first passive sonar encounter detection model. In this limit, the time between decisions approaches zero. However, in this limit the integration time that determines the recognition differential is not equal to δt and it does not approach zero. It is, in effect, chosen by the user of the model through the user's choice of the value for the recognition differential.

For an encounter that begins at t1 and ends at t2 and for which SE(t) is unimodal, it is shown in Reference 10 that for this limit, Equation 4 has the following form:

(9) $p = 1 - [1 - p(t_c)] \exp[-(1/\tau) \int_{t1}^{t2} p(t)\, dt]$

where $p(t) = \Phi[SE(t)/\sigma]$ and where now t_c is the encounter time such that $SE(t_c)$ is greater than equal to $SE(t)$ for any other encounter time t and $t2 \geq t_c$.

The Third Passive Sonar Encounter Detection Model: This model describes an encounter between a target and a passive sonar detection system in which detection occurs at a time t if the average value of the square of the continuously observed signal-to-noise ratio over a time interval of length u from time $t-u$ to time t is greater than or equal to the square of the signal-to-noise ratio that determines the recognition differential for that time interval. With $R(s)$ the random signal-to-noise ratio at a time s and $R_m(u)$ the random signal-to-noise ratio that determines the random recognition differential for the time interval u (the integration time), detection at time t is the following event:

(10) $$(1/u) \int_{t-u}^{t} [R(s)/R(u)m]^2 \, ds \geq 1.$$

The random integrand in the expression that defines the event is related to the random signal excess at the time s for an integration time u. The relation is:

(11) $$10 \log [R(s)/R_m(u)]^2 = 2[SE(s;u) + X(s)]$$

where $SE(s;u)$ is the expected value of the signal excess at a time s for an integration time u and $X(s)$ is the random component of the signal excess at the time s. In the model, $X(s)$ is determined by a lambda sigma jump process and $SE(s;u)$ is determined by an expected value sonar equation with a recognition differential: $RD(u) = 10 \log r_m(u)$. Here, $r_m(u)$

is the expected value of the signal-to-noise ratio that gives a probability of detection equal to .5 for an integration time u and a specified probability of false alarm p_f. With the signal detection process described by a Case II signal detection model, the detection index necessary to give the required operating point $(p_f,.5)$ is related to the integration time u and the signal-to-noise ratio $r_m(u)$ by:

(12) $d = u\cdot(BW)[r_m(u)]^2$

where BW is the bandwidth of the receiver. For a spectrum analyzer, BW would be the bandwidth corresponding to a given frequency resolution and d would be the detection index required in order to be at the operating point $(p_f,.5)$ for a signal that was contained within a bandwidth BW. Since d is fixed, because of Equation 12:

(13) $RD(u) = 5\log(t_0/u) + RD(t_0)$

where t_0 is a reference integration time. By using Equation 13, Equation 10 becomes:

(14) $$\int_{t-u}^{t} 10^{(1/5)[X(s) + SE(s;t_0) - 5\log(t_0)]}\, ds \geq 1$$

after moving the time origin so that $t \geq 0$. In this equation, $u = t$ for $t < t_0$ and $u = t_0$ for $t \geq t_0$ and $SE(s;t_0)$ is the expected value of the signal excess at the time s for an integration time t_0. In an encounter, detection occurs the first time that Relation 14 is satisfied.

As is pointed out in Reference 11, the appeal of the Third Passive Sonar Encounter Model relative to the Second and First Passive Sonar Encounter Detection Models is that it appears to more closely describe the detection process in passive sonar detection systems that display their processed data to an operator in a continuous manner. However, results reported in Reference 12 indicate that the difference between the three models may not be significant in some types of encounters.

VIII. Straight Line Encounters

Suppose a target's detectability depends on its range from a detection system and that the probability of detection is effectively zero beyond a range r_m for any target azimuth. In this report, an encounter between the target and the detection system is the event that the range between the target and the detection system is less than or equal to r_m. In addition, suppose r_m is small enough so that when the target and the detection system are having an encounter they can be considered to be moving on planes parallel to a tangent plane to the earth's surface at some point in their vicinity. If this is the case, then while the target and detection system maintain a constant course and speed during an encounter, the encounter is called a straight line encounter.

A straight line encounter can be described in terms of a two dimensional rectangular coordinate system whose plane is parallel to the tangent plane to the earth. If the coordinate system is stationary relative to the detection system with the detection system located at the origin and is oriented so that the target's motion is parallel to the y-axis and is in the positive y-direction, then the target's x-coordinate during a straight line encounter will be constant. The constant is equal to the target's horizontal range at the closest point of approach (CPA) on the straight line track on which the target is moving relative to the detection system during the encounter. This range is called the target's lateral range.

A complete straight line encounter is one in which a target begins an encounter on a straight track at a point on the relative track whose range from a detection system is equal to r_m, continues to CPA and then continues to a point on the relative track whose range from the detection system is again equal to r_m. Let $p(x)$ be the cumulative probability that a target is detected by a detection system in a complete straight line encounter in which the target's lateral range is x. Then the function $\mathbf{p(x)}$ defines what is called a **lateral range curve or lateral range function.**

Let p be the probability that a target is detected during a complete straight line encounter. If the lateral range of a target in a straight line encounter is assumed to be a continuous random variable X with a uniform distribution that extends at least from $-r_m$ to r_m where r_m is the maximum detection range, then the probability that a target will be detected during a complete straight line encounter is:

(15) $$p = (1/2r_m) \int_{-\infty}^{\infty} p(x)dx$$

where the limits of integration can be used since the value of $p(x)$ is zero for $x \le -r_m$ and for $x \ge r_m$. The quantity:

(16) $$W = \int_{-\infty}^{\infty} p(x)dx$$

is called the **sweep width**. Using the definition of W given by Equation 16, the probability that a target will be detected during a complete straight line encounter under the conditions that determine Equation 15 is $p = (1/2r_m)\cdot W$.

IX. Two Intermittent Signal Encounter Models

In the intermittent signal encounter models that are described in this section, an encounter is a complete straight line encounter and during an encounter a target emits a signal at various times. Two cases are considered: In the first case, the signals are emitted periodically, the signals are of length δt and the time between signals is τ where $\tau > \delta t$. In the second case, the signals are instantaneous and the signals are emitted at times determined by a Poisson process for which the expected time between signals is equal to τ. In the model, the detectability of a target signal depends on a target's horizontal range from a detection system, but on no other factors. If a signal is emitted while the target is within a range r, it will be detected. For a continuous signal, the lateral range function of a detection system for a target is: $p(x) = 1$ for $|x| \leq r$ and $p(x) = 0$ for $|x| > r$ where the horizontal range r is determined by the characteristics of the detection system and the target. The geometry for an encounter is shown in Figure 1 below.

For an intermittent signal, the length of a target's track relative to a detection system where if a signal is emitted it will be detected is $2(r^2 - x^2)^{\frac{1}{2}} + w \cdot \delta t$ with w the speed of the target relative to the detection system. So, a target's exposure time during an encounter is $(2/w)(r^2 - x^2)^{\frac{1}{2}} + \delta t$.

For a periodic signal, there are two cases. In the first case, $r \geq w \cdot (\tau - \delta t)/2$. In this case, the exposure results in the following lateral range function:

$$
\begin{aligned}
&p(x) = 0 \quad \text{for} \quad |x| > r \\
(17) \qquad &p(x) = 1 \quad \text{for} \quad |x| < \{r^2 - [w \cdot (\tau - \delta t)/2]^2\}^{\frac{1}{2}} \\
&p(x) = (2/w \cdot \tau)(r^2 - x^2)^{\frac{1}{2}} + \delta t/\tau \quad \text{otherwise}
\end{aligned}
$$

In the second case, $r < w \cdot (\tau - \delta t)/2$. In this case, the middle equality in Equation 17 does not apply.

For a signal that is instantaneous and whose emission times are determined by a Poisson process, the exposure results in the following lateral range function:

$$
\begin{aligned}
(18) \qquad &p(x) = 1 - \exp\{-[(2/w \cdot \tau)(r^2 - x^2)^{\frac{1}{2}}]\} \quad \text{for} \quad |x| \leq r \\
&p(x) = 0 \quad \text{for} \quad |x| > r
\end{aligned}
$$

for all values of r. A case in which the signal length δt is greater than zero is discussed below.

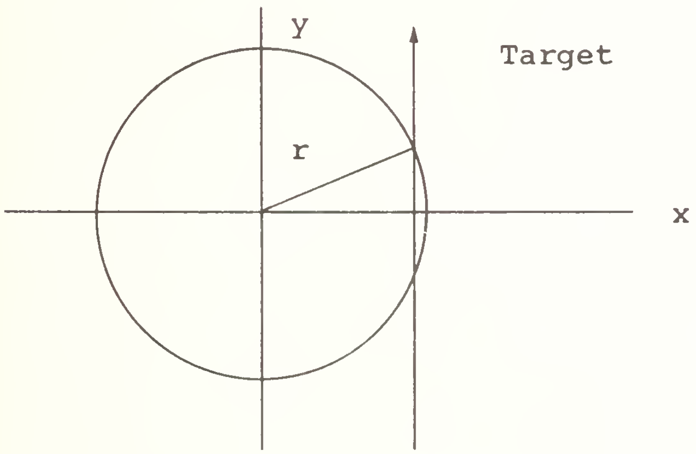

Figure 3. The encounter geometry for the two intermittent signal models described above.

For signals whose emission times are determined by a Poisson process and whose length is δt, signals can overlap. If this is allowed, then Equation 18 can be modified to describe this case by adding $\delta t/\tau$ to the term in the exponent of Equation 18 that is within the square brackets. In particular, note that this modified Equation 18 can be approximated by the bottom equality in Equation 17 when $(2/w\cdot\tau)(r^2 - x^2)^{\frac{1}{2}} + \delta t/\tau << 1$. This implies that when the expected time τ between signals is large relative to the exposure time $(2/w)(r^2 - x^2)^{\frac{1}{2}} + \delta t$ the periodic emission model and the random emission model are effectively equivalent.

X. A Random Search Model

A search of a region in which a target moves on a track that consists of a number of straight segments placed in such a way that in a limiting sense every section of the region is equally likely to be searched on a segment is referred to as a random search in Reference 6. Representation of a search region with the track segments that could be imagined to be the tracks of a random search are shown in Figure 4.

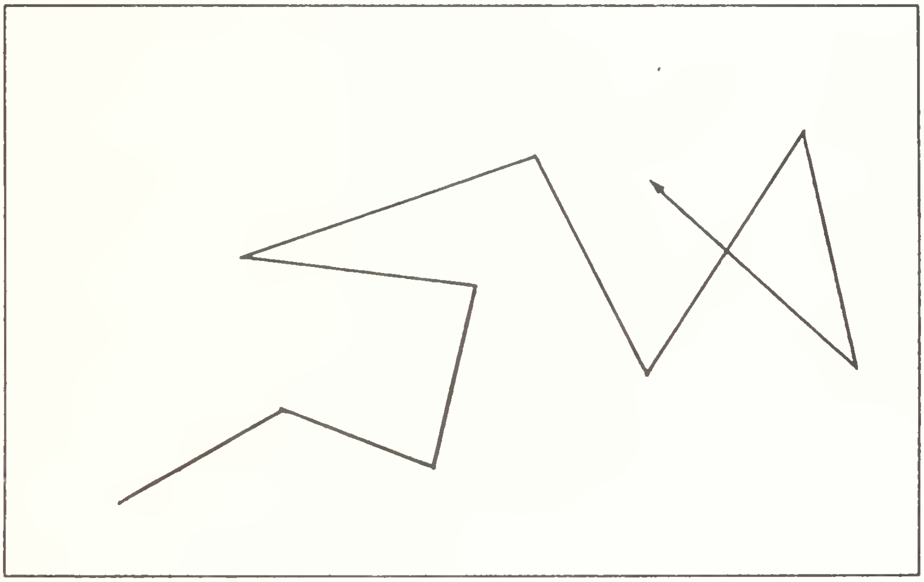

Figure 4. A search region and a track that could be described as a random search track.

Two developments of a model to describe this kind of search are contained in this section. The first development is based on the following conditions: 1. A target is at a fixed position within a defined search region. 2. A searcher's track is a sequence of straight line segments that are within the search region. 3. The searcher's detection system is such that while on a track segment, a rectangle is searched that is contained within the search region, is of length equal to the length of the track segment and is oriented so that its long axis is parallel to the track segment. 4. The probability that the searcher's detection system will detect a target while on a track segment with a search rectangle that does not include the target is zero. The probability that the searcher's detection system will detect a target while on a track segment with a search rectangle that includes the target is $p(x)$ where x is the target's lateral range for the track segment and $p(x)$ is the lateral range curve for a complete straight line encounter with this lateral range. A representation of a search rectangle is shown in Figure 5 below. 5. The track segments are located in such a way that the event that the target is within the search rectangle associated with a track segment is independent of the event that the target in the search rectangle associated with any other track segment. And the probability of the event is equal to the ratio of the area of the search rectangle to the area of the search region and, given a target is within a search rectangle, its position is uniformly distributed over the rectangle.

Condition 4 implies that the random search model is based on the concept of a complete straight line encounter. The definition of an encounter that is intended here is that given in Section VI. This implies that in the random search model the time to resolve a false alarm is zero. However, for the model, p_d and p_f are considered to be determined by some criterion such that p_f is less than one. Consequently, although the time to resolve a false alarm is zero in the model, the cost associated with a false alarm is not zero. (A concise model that accounts for the time to resolve false alarms is described in Reference 13.) Condition 4 also implies that when a searcher is on a track segment with a search rectangle that contains a target, the encounter is a complete straight line encounter. And Condition 5, which can be considered to specify a random arrangement of the track segments, implies that when this is the case, for the complete straight line encounter, the target's lateral range is a random variable that is uniformly distributed between $-b/2$ and $b/2$ where b is width of the search rectangle (the dimension of the rectangle perpendicular to the associated track segment).

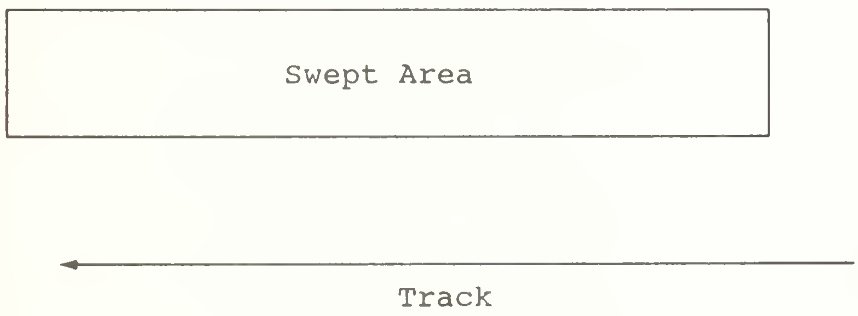

Figure 5. A track segment and its associated search rectangle. that could correspond to a search with an aircraft mounted infrared detection system.

Based on the above considerations, the probability that a target will be detected while a searcher is on a track segment with an associated search rectangle that contains the target is given by:

$$\int_{-\infty}^{\infty} p(x) f_X(x) dx = W/b \tag{19}$$

where $f_X(x) = 1/2$ for $-b/2 \le x \le b/2$ and $f_X(x) = 0$ and $p(x) = 0$ otherwise. Note that the left side of Equation 19 applies to any complete straight line encounter in which the target's lateral range for the encounter is considered to be a random variable with a distribution determined by the probability density function $f_X(x)$. If it is not given that the target is within the search rectangle associated with a track segment, then the unconditional probability that the target will be detected on the track segment is given by: $(W/b)(\delta A/A)$ where δA is the area of the search rectangle associated with the track segment and A is the area of the search region. With l the length of the rectangle, $\delta A = b \cdot l$ and the probability becomes: $(W \cdot l)/A$. Then, since the event that the target will be in the search rectangle of a track segment is independent of the event that it will be in the search rectangle of any other track segment, the probability p that a random search consisting of m track segments will detect the target is given by:
$1 - [1 - (W \cdot l_1)/A][1 - (W \cdot l_2)/A] \cdots [1 - (W \cdot l_n)/A]$ where l_i is the length of the i^{th} track segment. The probability is also given by: $p = 1 - \exp\{\Sigma \ln[1 - (W \cdot l_i)/A]\}$. If for

$i = 1,2, \cdots ,n$ $(W \cdot l_i)/A << 1$, then this expression can be approximated by:

(20) $$p = 1 - \exp[-(W \cdot l)/A]$$

where the sum index $i = 1,2, \cdots ,m$ and $l = \Sigma\, l_i$ is the track length of the search. Equation 20 is known as the **random search formula.**

The second development is based on the following condition: The detection rate of a searcher's detection system for a target is given by: $\tau(t) = [W \cdot v(t)]/A$ where $\tau(t)$ is the continuous detection rate function in Equation 5. Using the value for $\tau(t)$ and Equation 5, the random search formula given by Equation 18 is easily found.

The advantage of the method used in the second development of the random search formula over the method used in the first development is that, in general, it can be more easily used to determine the probability of detection for a search that can be defined in terms of a detection rate. Reference 14 contains an example of an application of Equation 5 to a case where the search region expands with time.

XI. Ladder and Barrier Search Models

In some barrier searches, the barrier search track is a ladder search track relative to a reference system that moves with the target. This fact is used in the barrier search model development that follows the two ladder search model developments below. The first ladder search model is referred to as an ideal ladder search model because of the idealizations that are involved in its description of a ladder search. The second ladder search model is referred to as a degraded ladder search. It can be considered to describe a ladder search track in which navigational errors result in omissions and overlaps in coverage.

An Ideal Ladder Search Model: The model is based on the following conditions: 1. A ladder search region is a rectangle that contains a fixed target. 2. During a search of the region, the searcher's detection system moves on a set of m parallel track segments of length b separated by a distance s. 3. As the detection system moves along a track segment, it searches a rectangular strip of length b and width s within the search region. 4. The m rectangular strips that correspond to the m track segments completely cover the ladder search region with no overlap. 5. If a target is within the rectangular strip corresponding to a track segment, then there will be a complete straight line encounter between the target and the detection system when the detection system moves along the track segment and the lateral range of the encounter will be uniformly distributed across the width of the strip. If the target is not in the rectangular strip, then there will not be an encounter and the probability that the target will be detected while the detection system is on the track segment is zero.

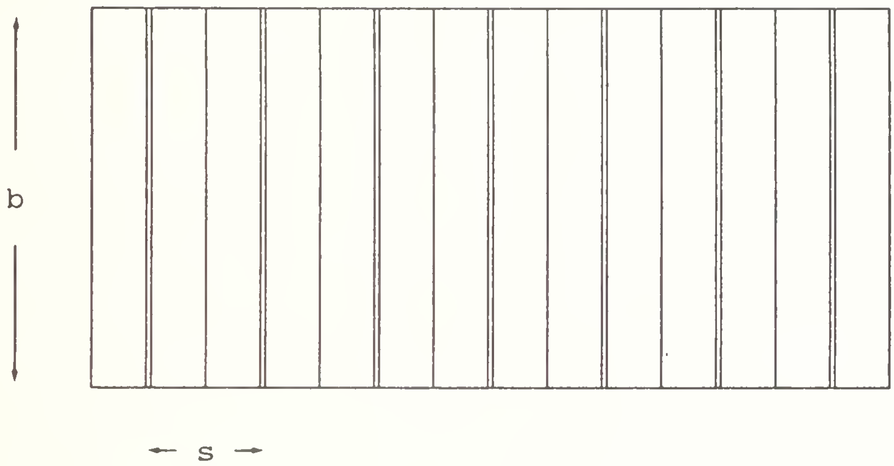

Figure 6. A schematic representation of a ladder search geometry for a case in which the ladder search track segments are superimposed on and bisect their corresponding rectangular strips.

Since targets outside of the rectangular strip that corresponds to a track segment cannot be detected while a detection system is on the track segment because of Condition 5, in the model, the sweep width W of a searcher's detection system must satisfy the relation $W \leq s$. In particular, $W = s$ only holds when the detection system detects a target that is in a rectangular strip with probability one for any target lateral range. This kind of detection system is sometimes referred to as a cookie cutter detection system. However, this terminology can be misleading since it suggests the detection system detects equally well for all azimuths. But this is not a requirement on the system in order that $W = s$.

The ideal ladder search model implies that if the conditions of the model are satisfied, then the probability p that a target will be detected by a an ideal ladder search is given by:

(21) $$p = W/s$$

where $W/s \leq 1$. The quantity W/s is called the coverage factor in this case.

A Degraded Ladder Search Model: The above model implies perfect navigation in addition to other idealizations. A model of a ladder search is given in Reference 6 that could be used for cases in which this is a poor assumption. The model which is referred to here as a degraded ladder search model can be considered to describe navigational inaccuracies in terms of omissions and overlaps of the rectangular strips. It can be developed as follows: Consider a random search in the ladder search region whose track length is equal to the search track length required to complete an ideal ladder search, that is, a track length $l = m \cdot b$. The degraded ladder search model describes the result of omissions and overlaps in a ladder search to be such that the probability of detection for this random search is equal to the probability of detection for the degraded ladder search. Consequently, since the area of the ladder search region is $m \cdot s \cdot b$, for the degraded ladder search model:

(22) $$p = 1 - \exp(-W/s).$$

Here, the requirement that the coverage factor $W/s \leq 1$ for Equation 21 can be relaxed. However, it should still be considered as an approximate condition.

The condition that the target be fixed within the rectangular search region is critical to both Equation 21 and Equation 22. However, these results are also applicable to a search for a moving target under the conditions that are described next.

A Barrier Search Model: A target moves with a constant course and a constant speed u. Both the target's course and the target's speed are known by a searcher. The searcher establishes a barrier of width b that is perpendicular to the target's track and moves on the barrier with a speed $v > u$. The barrier is designed so that in a reference system relative to the target the barrier search is a ladder search that satisfies the conditions for a ladder search that are given above. There are two cases to consider: 1. The barrier is established in front of the target. 2. The barrier is established behind the target.

From the search geometry for a barrier established in front of the target, it can be seen from Figure 7 below that $\theta = \sin^{-1}(u/v)$ and $d = v \cdot \tau$ where $\tau = s/(v + u)$ is the time to move from one search leg to the next. The angle θ and the perpendicular distance d that depend on u, v and s, and the width of the barrier b are the quantities that are required in order to establish the barrier operationally.

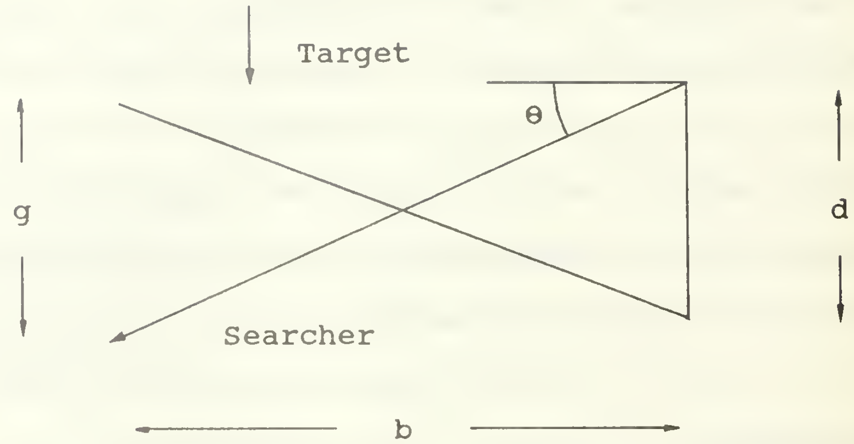

Figure 7. A barrier search track shown for a barrier established in front of the target. The track is shown in a reference system fixed relative to the earth.

For a barrier that is established in front of a target, one of three barrier types will result. A barrier's type is determined by the relation of the distance d to the distance $g = ut$ where the time $t = b/(v^2 - u^2)^{\frac{1}{2}}$ is the time to complete a search leg (cross the barrier). The barrier type is determined as follows: 1. For $g < d$, the barrier is an advancing barrier. 2. For $g = d$, the barrier is a stationary barrier. 3. For $g > d$, the barrier is a retreating barrier.

For a barrier established behind the target, there is only one barrier type and it is called an overtaking barrier. For an overtaking barrier, $\theta = \sin^{-1}(u/v)$ as for a barrier established in front of the target. But, for an overtaking barrier, $\tau = s/(v - u)$ and $d = v \cdot s/(v - u)$.

Given the target crosses the barrier, the probability of detection for an ideal barrier search is given by Equation 21 and the probability for a degraded barrier search is given by Equation 22 where the terminology refers to the nature of the ladder search in the reference system moving with the target. A discussion of an application of these two equations to a search for a magnetic anomaly target is given in Reference 15.

XII. A Target State Estimation Procedure

A target state estimation procedure based on bearing observations is developed in this section that generates point estimates of a target's position and velocity vector coordinates in a rectangular coordinate system. The procedure is based on a model in which bearing errors are unknown and are not determined by random variables with known distributions. Because of this, confidence regions for the estimates are not generated by the procedure. However, for a moving target, it illustrates some general characteristics of bearings only **target motion analysis (TMA)**.

The target state estimation procedure model is defined by: 1. The target moves in a plane with a constant but unknown course and speed. 2. Observations of the target are made from known positions at known times. 3. The observations provide only target bearings with unknown errors. 4. Target position estimates $[x_t(i), y_t(i)]$ for $i = 1, 2, \cdots, n$ and target velocity component estimates u_x and u_y are chosen so that the sum $S = \Sigma\, d_i^2$ is a minimum where d_i is the algebraic distance between the estimated position and the bearing line corresponding to the observed bearing.

Because of the requirement that the target move with constant course and speed during the encounter, the number of independent estimates is reduced from $2m$ to 4: $x_t(1)$, $y_t(1)$, u_x and u_y. These four estimates constitute a set of target state parameters that define a target's motion by the relations:

$x_t(i) = x_t(1) + u_x \cdot (t_i - t_1)$ and $y_t(i) = y_t(1) + u_y \cdot (t_i - t_1)$. The geometry on which the procedure is based is shown in Figure 8 where $d_i = [x_t(i) - x_o(i)] \cdot \cos \tilde{\theta}_i - [y_t(i) - y_o(i)] \cdot \sin \tilde{\theta}_i$.

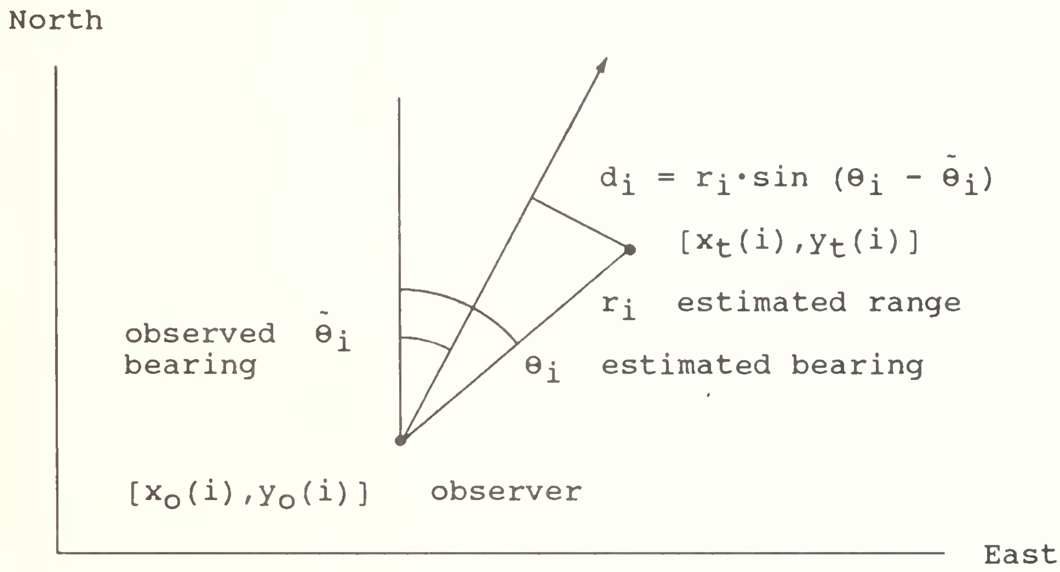

Figure 8. The geometry of the target motion analysis model.

To determine estimates of the target state parameters, take the partial derivative of S with respect to each of them. Then set the four partial derivatives equal to zero. This creates four linear equations in the four unknown estimates. In matrix notation, the equations are: $AX = B$ where the elements of X are $x_t(1)$, $y_t(1)$, u_x, and u_y. A necessary condition for a unique solution for X is that $m \geq 4$. Otherwise, the determinant of A will be equal to zero.

Now, suppose the observations are at positions and times that correspond to the positions and times of an observer moving on some constant course at some constant speed (including zero speed). In this case, the observation position coordinates are related by the following equations: $x_o(i) = x_o(1) + v_x(t_i - t_1)$ and $y_o(i) = y_o(1) + v_y(t_i - t_1)$ where v_x and v_y are the required velocity components of the observer. Using these equations of motion, the matrix equation $AX = B$ can be transformed to the matrix equation $AX' = 0$ where the elements x_{i1} of the matrix X are related to the elements of the matrix A' by the equations: $x'_{11} = x_t(1) - x_o(1)$, $x'_{21} = y_t(1) - y_o(1)$, $x'_{31} = u_x - v_x$ and $x'_{41} = u_y - v_y$.

Since the equations represented by $AX' = 0$ are homogenous, the equations do not have unique solutions and consequently the equations represented by $AX = B$ do not either. However, if there is at least one observation whose time and position is not determined by the above equations of motion, then the transformation from X to X' cannot be made, and in general a unique solution for X can be found. If the observations are made from a platform that is moving with a constant course and speed, this condition can be achieved by either changing the course, the speed or both prior to completing the observations.

Estimation models that describe bearing error in terms of a random variable with a specified distribution provide a basis for determining a confidence region to associate with the point estimates. A model is developed in Reference 16 that does this, however, it requires that either observations of a target bearing be made from two or more locations simultaneously or that the target be stationary relative to the observation points.

XIII. Position Distributions That Change with Motion

In this section, target motion models are discussed that provide a basis for determining position distributions that change with time. In the first model, a target moves in a plane with a constant course and speed. In the second model, a target moves in a plane with a constant course and speed during a time step and at the end of the time step changes either its course or speed or both.

Position Distributions based on the First Motion Model: Two cases of this model are considered that provide the basis for determining a position distribution analytically. In both cases, the target's position coordinates in a rectangular coordinate system are determined by a circular normal distribution at time zero. And, in both cases, during the motion the target's velocity does not change.

The First Case: In this case, the targets's coordinates $X(t)$ and $Y(t)$ at time t are determined by the vector sum with components: $X(t) = X(0) + U_x \cdot t$ and $Y(t) = Y(0) + U_y \cdot t$ where $X(0)$ and $Y(0)$ are the target's coordinates at time zero and U_x and U_y are the target's velocity components. The coordinates $X(0)$ and $Y(0)$ are determined by a circular normal distribution whose mean vector is $(0,0)$ and whose standard deviation is σ. And the velocity components U_x and U_y are determined by a circular normal distribution whose mean vector is $(\hat{u}_x, \hat{u}_y)$ and whose standard deviation is σ_u. The random vectors $[X(0), Y(0)]$ and $(U_x \cdot t, U_y \cdot t)$ are independent and both are normal random vectors. Since they are normal random vectors, the random vector $[X(t),Y(t)]$ is also a normal random vector. And, since they are independent, its characteristic function is equal to the product of their characteristic functions. This implies that $[X(t),Y(t)]$ has a circular normal distribution whose mean vector is $(\hat{u}_x \cdot t, \hat{u}_y \cdot t)$ and whose variance is $(\sigma^2 + \sigma_u^2)$. For a more complete discussion of the basis for the above argument, see Reference 7.

The Second Case: The procedure that is used to determine the distribution of the random vector $[X(t),Y(t)]$ for the second case can also be used to determine it for the first case. It is based on the following consideration: In general, the joint density function of a target's coordinates at some time t greater than a zero of time is determined as follows:

$$(23) \quad f_{X(t),(t)}(x,y;t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X(0),Y(0)}(r,s;0)\, f_{U_x,U_y}(v,w)\, dv dw$$

where $r = x - v \cdot t$ and $s = y - w \cdot t$ with $v = u_x$ and $w = u_y$. Equation 23 can be interpreted as follows: After multiplying the right-hand side of Equation 23 by $\delta r \delta s$, in the limit, it is the sum over all pairs of values (v,w) of the product of the probability that the target's velocity components (v,w) are in an element $\delta v \delta w$ and the probability that the target's position coordinates (r,s) at time zero are in an element of area $\delta r \delta s$. This is the position such that the targets's position coordinates (x,y) at time t will be in an element of area $\delta x \delta y$ given that the target's velocity is (v,w). And, in the limit, this is equal to the left side of Equation 23 after multiplying the left side by $\delta x \delta y$. But, since $\delta r \delta s$ is equal to $\delta x \delta y$ in the limit, Equation 23 results. In effect, $\delta r \delta s$ is translated from (r,s) to (x,y) without rotation or distortion.

In the first case, the joint distribution of the of the velocity components (U_x,U_y) is circular normal with mean vector $(\hat{u}_x,\hat{u}_y)$ and standard deviation σ_u and the joint distribution of $[X(0),Y(0)]$ is circular normal with mean vector $(0,0)$ and standard deviation σ. In the second case, the target's speed u is known but its course is equally likely to have any value between 0 and 2π, so $U_x = u \cdot \sin \Phi$ and $U_y = u \cdot \cos \Phi$ where Φ is a uniform random variable and $0 \le \Phi < 2\pi$. In this case, only the single random variable Φ is required to determine the target's velocity. The integral of Equation 23 in polar coordinates is a single integral over Φ and the integrand is $(1/2\pi\sigma^2) \cdot \exp[-(r^2 + s^2)/2\sigma^2] \cdot (1/2\pi)$ where $r = x - u \cdot t \cdot \sin \phi$ and $s = y - u \cdot t \cdot \cos \phi$. The result of the integration is:

(24) $(1/2\pi\sigma^2) \cdot \exp\{-[x^2 + y^2 + (u \cdot t)^2]/2\sigma^2\} I_0[(x^2 + y^2)^{\frac{1}{2}} u \cdot t/\sigma^2]$

where I_0 indicates the hyperbolic Bessel function of zeroth order. Expression 24 is plotted in Reference 6 for several values of t. There, a single variable is substituted for the distance of the target from the origin $(x^2 + y^2)^{\frac{1}{2}}$. It is measured in units of σ and the time is measured in units of σ/u. The plot of the expression shows a limiting characteristic of the distribution that can be indicated as follows: Multiply and divide the resulting expression by $\exp(-z \cdot u \cdot t/\sigma^2)$ where

$z = (x^2 + y^2)^{\frac{1}{2}}$, the target's range from the origin. The result of the operation can be expressed in the following form:

$$(25) \quad 1/(2\pi\sigma^2)\exp\{-[1/(2\sigma^2)](z - u \cdot t)^2\}I_0(z \cdot u \cdot t/\sigma^2)\exp(-r \cdot u \cdot t/\sigma^2)$$

where the second factor in Expression 25 is proportional to the density function of a normal random variable Z whose mean is $u \cdot t$ and whose standard deviation is σ. The last two factors are such that their product changes slowly relative to the second factor. The consequence of this is that a plot of Expression 25 against z in units of σ for values of t in units of σ/u has the appearance of the plot of a normal density function for values of t greater than four in units of σ/u. In particular, this indicates that $u \cdot t$ in the exponent of the second factor in Expression 25 can be viewed as the radius of an average furthest on circle.

A target's rectangular coordinates and its range and bearing from the origin are related by: $X = R \sin \Theta$ and $Y = R \cos \Theta$. By using these equations, Expression 24 can be transformed to:

$$(26) \quad (1/2\pi)(r/2\pi\sigma^2)\exp\{-[r^2 + (ut)^2]/2\sigma^2\}I_0(rut/\sigma^2)$$

which is the joint density function of the random variables R and Θ. The marginal density function of R can be obtained from Expression 26 by integrating this joint density function over the possible values of Θ which in radians is over the interval 0 to 2π. Consequently, the marginal density for R can be obtained from Expression 26 by multiplying it by 2π. Tabulated values of the cumulative distribution function $F_R(r;t)$

for the marginal distribution of R for the second case are listed in Reference 17.

Position Distributions Based on the Second Motion Model: In the second motion model a target's motion is not restricted to a constant course and speed throughout the motion. At the end of a time step, a target's course or speed or both can change. In general, this leads to a position distribution that can not be described analytically. And, in general, a monte carlo simulation method is required in order to determine a position distribution that is based on such a model. As an example of a case in which the distribution can be described analytically, see Reference 18. As an example of a case in which it can not, suppose a target's initial position is described in terms of a number assigned to a subregion in the xy-plane where the number assigned represents the probability that the subregion contains the target at an initial time. In addition, suppose for each subregion a course and speed distribution is determined by assigning numbers to course and speed pairs where a number represents the probability the target will have the course and speed at the initial time given it is in the subregion at that time. Next suppose for each course an speed pair there is a time distribution that determines the duration of the course and speed pair and that the time distribution is determined by a number assigned to each discrete time point where the number represents the probability that the targets course and speed pair will be determined by a new course and speed distribution given it has not been determined at an earlier time point. By extending this kind of procedure and then implementing it in a monte carlo simulation, one can generate complex position distributions that describe a target's position at discrete time points in terms of probabilities assigned to subregions of a region in which the target.

XIV. Position Distributions That Change with Search

Suppose a target's position at some time is described by a position distribution. Now suppose information becomes available that a search has been conducted for the target and that the target has not been detected. Or suppose the information is that the target has been detected. In the first case, negative information is available that can be used to modify the position distribution. In the second case, positive information is available that can be used to modify the position distribution. In both of the cases, a target's position distribution is assumed to be specified in terms of a set of discrete probabilities where each probability corresponds to a subregion of the region that contains the target and each is the probability that the target is in that subregion.

Position Distributions and Negative Information: For a region that contains a target and consists of n subregions, let the event S_i = {the target is in the i^{th} subregion}. And let the event $\overline{C}$ = {no detection during a search of the region}. Then, given an unsuccessful search in the region, the targets's position distribution can be modified as follows:

(26) $P(S_i|\overline{C}) = P(\overline{C}|S_i)P(S_i)/P(\overline{C})$

where $i = 1,2, \cdots ,n$ and $P(\overline{C}) = \Sigma\, P(\overline{C}|S_j)P(S_j)$ with the sum from $j = 1,2, \cdots ,n$. Note that Equation 26 can be obtained by using Bayes theorem. To illustrate how Equation 26 might be used, suppose that a search in a subregion is considered to be a random search and that the sweep width of detection system against the target depends on subregion being searched. For this case, let A_i be the area of the i^{th} subregion and let W_i be the sweep width in that subregion. Then, given an unsuccessful search, $P(\overline{C}|S_i) = \exp(-W_i \cdot l_i/A_i)$ where l_i is the track length of the searcher in the i^{th} subregion. Given values for $P(S_i)$, δ_i and l_i for $i = 1,2, \cdots ,n$ a position distribution can be determined that has been modified by the negative information.

Position Distributions and Positive Information: As above, let S_i the event that a target is in the i^{th} subregion of a region that contains the target. However, in the case of positive information, instead of the event $\overline{C}$ occurring, the event $C = \{\text{a contact}\}$ occurs. This event is the union of two mutually exclusive and exhaustive events: $T = \{\text{a true contact}\}$ and $F = \{\text{a false contact}\}$. Relative to the Venn diagram of Figure 1, for a subregion i, S_i corresponds to H_1, C corresponds to D_1, T corresponds to $(D_1 \cap H_1)$ and F corresponds to $(D_1 \cap H_0)$. For generality, suppose true contacts do not localize a target to a single subregion. Then after a search of a region that has resulted in a contact, the target's position distribution can be modified as follows:

(27) $P(S_i|C) = P(S_i|T)\ P(T|C) \ + \ P(S_i|F)\ P(F|C)$

where $i = 1,2, \cdots ,n$. Equation 27 results since $T = T \cap C$, $F = F \cap C$ and $P(S_i|C) = \{P[(S_i \cap T) + P(S_i \cap F)]\}/P(C)$. The probability $p = P(T|C)$ has been called the credibility of the contact. In terms of p Equation 27 becomes:

(28) $P(S_i|T) = P(S_i|T)p + P(S_i|F)(1 - p)$.

For the model, $P(S_i|T)$ can be determined by the coverage characteristics of the detection system that is used to make the contact. In particular, $P(S_i|T) = P(T|S_i)P(S_i)/P(T)$ where $P(T) = \Sigma P(T|S_j)P(S_j)$ and the sum index $j = 1,2, \cdots ,n$. Then, using the correspondence between the events S_i and H_1 and the events T and $(D_1 \cap H_1)$ one can write $(p_d)_i = P(T|S_i)$ where $(p_d)_i$ is an average probability of detection over the i^{th} subregion. In one positive information model, $P(S_i|F) = P(S_i)$, the probability the target is in the i^{th} subregion before the contact information was available, and p is determined subjectively based on factors associated with a search. The probability determined by $1 - p$ has been called the false alarm probability. However, it is not the probability of a false alarm as it is defined in Section I. Since $1 - p = P(F|C)$, it corresponds to $P(H_0|D_1)$ not to $p_f = P(D_1|H_0)$.

XV. Search Models and Search Theory

Search theory provides a basis for determining optimal search plans for a target whose motion and location are determined within some bounds. Here, an optimal search plan is one for which the probability of finding a target within a given length of time is a maximum, the expected time to find a target is a minimum given the target is found or a search plan for which some other optimal search criterion is satisfied.

Search theory results are based on models of the search process. To the degree that a search model describes a search process, an optimal search plan for a target that is based on the search model should provide guidance for the development of an operationally feasible search plan. However, because of the limitations of analytical search models, an optimal search plan that is based on an analytical search model may give only initial guidance in this regard. The optimal search plans that are described below illustrate this. The search plans are based on the random search model. Because of this, the requirement on the location of search track segments is not realizable and the time to resolve false alarms is ignored.

Optimal search plans based on search models implemented through a monte carlo simulation are not considered here. However, with sufficient information, such plans have the potential of being both implementable and more optimal in a real sense than an optimal search plan based on an analytical search model.

Three Optimal Search Plans: The three optimal search plans differ through their definition of optimality. However, each one is based on the following search model: A target is fixed at some point in a region that consists of n subregions. A search in a subregion is a random search in the sense of the definition in Section X and a searchers sweep width there is a constant. In addition, a search of a subregion will not detect a target which is in another subregion. To determine a plan, let S_i = {the target is in subregion i} for $i = 1,2, \cdots ,n$ and let $p_i = P(S_i)$ be the prior probability that the target is in the i^{th} subregion. Let W_i be the sweep width in the i^{th} subregion. Let $\delta_i = A_i/W_i$ where A_i is the area of the i^{th} subregion and δ_i is the expected track length to find the target by a search of the i^{th} subregion given the target is in the i^{th} subregion, a characteristic length. The probability P that the target will be detected by a random search is given by:

$$P = \Sigma \, [1 - \exp(-l_i/\delta_i)] \cdot p_i \qquad (29)$$

where the sum index $i = 1,2, \cdots ,n$ and l_i is the track length of the search in the i^{th} subregion.

The first optimality criterion is: Choose l_i so that P is a maximum subject to the two constraints: 1. $l = \Sigma \, l_i$ and 2. $l_i \geq 0$ where the index $i = 1,2, \cdots ,n$. Determining this choice is a nonlinear optimization problem whose solution is given in Reference 19. It is:

$$(30) \qquad \begin{array}{ll} l_i^*/\delta_i = \ln(p_i/\delta_i) - L(k) & i = 1,2, \cdots , k \\ l_i^*/\delta_i = 0 & i = k+1,k+2, \cdots ,n \end{array}$$

where $L(k) = (1/\Sigma\delta_j)\cdot\Sigma[\delta_j\cdot\ln(p_j/\delta_j)] + 1/\Sigma\delta_j$ and the sum index $j = 1,2, \cdots ,k$, where the subregions are relabeled so that the following order relation holds: $p_1/\delta_1 > p_2/\delta_2 > \cdots > p_n/\delta_n$ and where k is chosen so that for $k+1$ the solution for l_{k+1} using $L(k+1)$ is either negative or zero.

The second optimality criterion is: Choose l_i so that P is a maximum subject to the two constraints: 1. $c = \Sigma\, c_i$ and 2. $c_i \geq 0$ where the index $i = 1,2, \cdots ,n$, $c_i = k_i\cdot l_i$ is the cost of the search in the i^{th} subregion and k_i is the cost per unit track length in that subregion. For this criterion, the solution to the corresponding nonlinear optimization problem can be obtained from Equation 30 by replacing δ_i by $\epsilon_i = k_i\cdot\delta_i$ and labeling the subregions so that $p_1/\epsilon_i > p_2/\epsilon_i > \cdots > p_n/\epsilon_i$. The basis for this can be seen by replacing l_i/δ_i by its equivalent c_i/ϵ_i in the exponential term in Equation 29.

The third optimality criterion is: Choose l_i so that the expected utility of finding the target is a maximum subject to the two constraints: 1. $l = \Sigma\, l_i$ and 2. $l_i \geq 0$ where the index $i = 1,2, \cdots ,n$. For this criterion, the solution to the corresponding nonlinear optimization problem can be obtained from Equation 30 by replacing p_i by q_i where $q_i = u_i\cdot p_i$ and u_i is the utility of finding the target given it is in the i^{th} subregion. And, in addition, labeling the subregions so that $q_1/\delta_1 > q_2/\delta_2 > \cdots > q_n/\delta_n$. The basis for this can be seen by multiplying the summation term in Equation 29 by u_i so that

Equation 29 gives the expected utility of finding the target with the search.

Equation 30 can be used to determine an order of search for the subregions which will effectively minimize the expected track length required to detect a target given it is detected. To do this, divide the available track length l into units small enough so that with a single unit only the 1^{st} subregion would be searched. Then allocate one unit to the search of the 1^{st} subregion. If the search is unsuccessful, determine the optimum allocation for two units. Then search with a second unit so that the first search with the first unit plus the second search with the second unit satisfy the optimum allocation for two units. If the search is unsuccessful, continue in this fashion until either the target is found or all the track length is expended. That this allocation order will effectively minimize the expected track length required to detect a target given it is detected can be argued as follows: Let L be the track length at detection, let l_u be a unit of track length and let n be the number of units. Then the value of the probability $P(L \leq i \cdot l_u)$ that the target will be detected on or before the i^{th} step of the search for the given allocation order will be greater than or equal to its value for any other allocation order with the same allocation step size. Since the value of $P(L \leq l)$ will be equal to its value for any other allocation order of the optimum allocation and since $P(L \leq i \cdot l_u | L \leq l) = P(L \leq i \cdot l_u)/P(L \leq l)$, the value of the distribution function $F_L(i \cdot l_u | L \leq l) = P(L \leq i \cdot l_u | L \leq l)$

will also be greater than or equal to its value for any other allocation order. This implies that the expected track length given detection $E(L|L \le 1) = \Sigma\ [1 - F_L(i \cdot l_u|L \le 1)]$ where the sum index $i = 1,2, \cdots ,n$ is effectively a minimum for the given allocation order. A search based on the optimum allocation given by Equation 30 and the given allocation order is equivalent to the following search prcedure: After an allocation of track length l_u and an unsuccessful search, new values for $P(S_i)$ are calculated using Equation 26 and then Equation 30 is used with these new values to determine the next optimum allocation. A discussion of this procedure is given in Reference 6. And an example of its application is given in Reference 20.

Equation 30 also defines an optimal search plan for a detection system that searches beams and can be described by Equation 28 by replacing l_i by t_i where t_i is the time the i^{th} beam is searched and by replacing δ_i by τ_i where τ_i is the expected time to detect the target by a search of the i^{th} beam given the target is in the i^{th} beam, a characteristic time.

For a more extensive discussion of search theory and its application to military operations research, see Reference 21.

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